

## EVOLUTION OF ROGUE WAVES AND BREATHER OPTICAL SOLITONS IN INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION

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**Abstract:** This paper reports results of soliton dynamics investigated by solving the inhomogeneous Schrödinger equation (INLSE), which involves terms describing defects mostly encountered in optical systems. Solutions to the INLSE yield breather and rogue solitons arising initially from smooth pulses, for which their evolution is explored by analysing the parameter space and applying external perturbation that is quasiperiodic in nature. We demonstrate that the solitons can be controlled by proper adjustment to both the inhomogeneity parameter and frequency of the externally applied signal, for which rogue and localised breather waves could be generated. Further, we argue that the resulting solitons stand robust against the uniformly distributed noise seeded into the system. The findings help reveal factors that impact the dynamics of such solitons and tailoring them for potential photonic applications.

**Keywords:** Optical soliton; rogue wave; nonlinear Schrödinger equation.

**Introduction:** Soliton and breather waves have increasingly become the subject matter of oriented research activities in diverse systems and one such system is a nonlinear optical medium<sup>1</sup>. Fundamental solutions of the nonlinear wave equation such as the nonlinear Schrödinger equation (NLSE), establish for the soliton and breather generation in systems like these<sup>2</sup>. The optical solitons, on one hand, have potential importance in the growing field of information and telecommunication technology owing to their ability to propagate long distances without abating. On the other hand, controlling and investigating factors affecting the dynamics of these optical solitons is of immense significance to properly tailor their properties for practical applications. In one dimension, the NLSE is

essentially integrable and yields soliton solutions<sup>3</sup>, and it has been receiving elaborate investigation with regards to constraints and conditions that would make the soliton wave controlled and robustly stand out for utilisation in optical transmission<sup>4</sup>. The  $(2D + 1)$  NLSE has recently been employed to understand and model the characteristic dynamics of directional localised waves<sup>5</sup>, which are commonly encountered in oceanography and simulate the occurrence of breather beams. The results help explore solitons in  $(2D + 1)$  manifold with the formation of large-amplitude rogue waves having specifically finite crest length in nonlinear dispersive media systems like the Bose-Einstein condensates, superfluids, space plasma and photonics.

The issue of controlling and stabilising soliton waves has been the focus of research articles in the field during the last few years. *Inc et al* have analytically reported solutions of optical solitons by examining the resonant nonlinear Schrödinger equation (R-NLSE) which describes the generation and propagation of soliton waves in optical fibres by

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Published on Web 01/10/2023, [www.ijsonline.org](http://www.ijsonline.org)

adopting a sine-Gordon algorithm, which proved to be an efficient integration scheme<sup>6</sup>. They introduced three types of nonlinear terms ascribed to the optical fibres and accordingly derived solutions depending on specific constraints, naturally imposed on the system under investigation that preserved the existence of bright, dark, and bright-dark solitons. The importance of sine-Gordon model to approach a condensed state system such as the extended 1D (long) Josephson junctions (LJJ) has been addressed in a plethora of publications. Most importantly, the evolution of fluxon soliton waves under weak perturbation has been investigated and explored in terms of the structural potential ascribed to the system, with emphasis on symmetry properties and dispersion, which remarkably contributed as control parameters to preserving the temporal evolution of magnetic solitons<sup>7,8,9</sup>. Recent research<sup>10</sup> has demonstrated the feasibility of boosting the performance of microresonators, which work as convertors of laser light into ultrashort pulses (dissipative Kerr solitons) propagating around the circumference of the resonator. This has been achieved by creating the so-called perfect soliton crystals PSCs. The soliton pulse train repetition is governed by the size of the microresonator, and the smaller the size the higher the repetition rate which may reach the terahertz frequency domain. This is of course promising for enhancing the performance of defect-free optical communication links by increasing the speed and precision of signals. The size of the microprocessor is limited to a few tens of microns, however the researchers successfully managed to maximise the number of dissipative Kerr solitons equally spaced inside the microresonator as much as they could. By doing so they were able of generating the PSC, boosting interferometry and coherent multiplication of power and repetition rate of the produced pulse train, and the dynamics of the PSC formation were accordingly investigated.

In addition, the inhomogeneity of NLSE i.e., when defects are existing in the system, has been receiving increasing interest to obtain a conclusive

view on how solitons evolve and how they are influenced under imposed specific constraints. Embarking on 1-fold Darboux transformation, *Young-Sheng et al* studied the inhomogeneous Schrödinger equation (INLSE) and explicitly reported analytical first-order solutions that involved deformed breather and rogue waves<sup>11</sup>. Their findings demonstrated that the defect (inhomogeneity) has a critical impact on the soliton wave reflected in a change in the height and background of the generated solitons. Furthermore, it could be possible to control the rogue wave characteristics by properly adjusting a specific physical parameter such as the inhomogeneity parameter and reducing the resulting rogue soliton wave to the corresponding NLSE one under certain conditions set on other parameters. Therefore, it seems reasonably quite significant to further explore and analyse the parameter space influencing the evolution of solitons and their localisation characteristics in inhomogeneous Schrödinger equation due to their paramount importance in several applications and particularly in photonics.

### Governing inhomogeneous Schrödinger equation Nonlinear Schrodinger equation (NLSE)

In a general form, the nonlinear Schrödinger equation can be expressed by

$$iu_t + u_{xx} + 2|u|^2u = 0 \quad (1)$$

where  $u(x,t)$  is the dependent variable,  $u_t$  and  $u_x$  denote partial derivatives to time and space respectively. As can be noticed  $u$  is complex-valued in this case contrary to the sine-Gordon or Korteweg- de Vries (KdV) equations<sup>12</sup>. For optical systems  $t$  and  $x$  are exchanged in equation (1) ( $u_{tt}$  and  $u_x$  are considered then), which in this case has one solution that owns resemblance to Peregrine soliton when applied to fibre optics as follows<sup>13</sup>

$$u(x,t) = \sqrt{P_0} \left[ 1 - \frac{4(1+2i\frac{x}{L_{nl}})}{1+4(\frac{t}{T_0})^2+4(\frac{x}{L_{nl}})^2} \right] e^{i\frac{x}{L_{nl}}} \quad (2)$$

$L_{nl}$  is a nonlinear length with  $P_0$  is the power of the continuous background, and  $T_0$  is a time duration given by  $T_0 = (L_{nl})^{1/2}$ . This optical soliton given by

solution (2) is shown in Fig. 1 and is reminiscent of a rogue wave.

### **Inhomogeneous NLSE**

In order to account for defects in a realistic physical medium, additional explicit terms ought to be included into Equation (1) that becomes inhomogeneous and thus has the form [6]

$$iu_t + u_{xx} + 2|u|^2u + iau + a^2x^2u = 0 \quad (3)$$

where  $a$  is a real nonuniformity number representing the system inhomogeneity and conspicuously when it is zero Equation (3) reduces to the normal NLSE Equation (2). The significance of Equation (3) is that it allows modelling breather waves and their evolution in systems with embedded nonuniformity. In this context, experimental findings indicate to the pivotal role played by system defects in generating and influencing optical rogue waves and the necessity to working out a control mechanism over these waves<sup>11,14</sup>. Although solutions and parametric control on shape and amplitude of soliton waves of a INLSE have been proposed<sup>11</sup>, there is no compelling evidence existing yet on how the shape and background of the produced soliton would be accounted for when there exists an external perturbation or noise is in play as well. This may also have relevance to signal instabilities and optical chaos in a nonlinear system for which solitons are generated for telecommunication purposes, and it is significant to explore how their dynamics are affected. Driven by these findings, the objective of this paper is to address and simulate this problem setting out from solutions to Equation (3) and show accordingly the evolving wave instability trends, taking into consideration externally applied signals to explore the soliton dynamics, and hence show how robust they would stand against perturbations set within the system.

### **Numerical modelling and results**

#### **Solitons in NLSE, $a = 0$**

Solutions to the normal NLSE have been obtained using the COMSOL Multiphysics modelling platform to solve the PDE Equation (1), considering one-dimensional domain in the range  $[-10, 10]$  to construct the computation mesh. Imposing a

continuous periodic condition, the initial values were as follows

$$\begin{aligned} u(x, 0) &= u_0 e^{(-x^2 - \frac{3}{2})} \\ u_t(x, 0) &= 0 \end{aligned} \quad (4)$$

where  $u(x, 0)$  accounts for an initial Gaussian wave at  $t = 0$  which was used to solve for subsequent time intervals by adopting a 0.01s time step. The Dirichlet boundary condition was  $u(x, t)|_{x=r} = 0.02$ . Fig. 2(a) shows snapshots of the produced soliton at selected times for  $u_0 = 6$  on the  $[-10:10]$  domain. As can be seen in Fig. 2(a), the wave evolves temporally, and it progressively exhibits changes in direction resulting in localised bright and dark solitons clearly seen in Fig. 2(b) and its XY projection 2(c) with manifestly rippling background.

#### **Solitons in INLSE, $a = 0.012$ (Rogue wave)**

Equation (3) which includes inhomogeneity terms has been numerically solved for  $a = 0.012$  taking the same initial conditions (4). Fig. 3 shows progression of a rogue wave in a sequence from left to right computed for the first 4 seconds. As can be seen, the background is nearly flat at beginning and producing a relatively small amplitude wave, then it gets higher in intensity over the course of time exhibiting distinctly weak wave packet background, as demonstrated in Figs. 3 (a-d). In order to check the robustness of the resulting rogue wave, solutions were obtained by considering uniformly distributed noise with random seed and zero mean, also increasing the noise amplitude from 1 to 2 and 7 as can be seen in Figs. 4 (a-c) respectively. As observed, the rogue wave stands eminent in (a) and (b) but gets obfuscated once the noise amplitude has marginally increased in (c). Furthermore, the rippled background already observed in Fig. (3) becomes noisy and gains higher amplitude that is mostly manifested in (c).

#### **Solitons in INLSE under perturbation**

In this section solutions of the INLSE are obtained for  $a = 0.012$  as well but taking external perturbation into account. The equation of concern thus can be expressed as

$$iu_t + u_{xx} + 2|u|^2u + iau + a^2x^2u = f(t) \quad (4)$$

where  $f(t)$  is the applied time-varying signal accounting for perturbation terms and is given by

$$f(t) = \sin(\omega_1 t) \cos(\omega_2 t) \quad (5)$$

for which  $\omega_1 = 0.056$ , and  $\omega_2 = 1$  are taken as the frequencies of the quasiperiodic signal. Solution of Equation (4) is depicted in Fig. 5, which remarkably demonstrates the generation of two solitons emanating from the evolution of one soliton wave as seen in (5a). This is also demonstrated in Fig. (5b) which shows 2-dimensional top view plot of the evolving waves. One significant observation one can make here is that the resulting two solitons are generated with manifestly large amplitude too. However, if the initial condition is represented as a Gaussian pulse wave of the form  $u(x, 0) = (6/\sqrt{2\pi}) \exp(-(x - x_0)^2/2)$  applied at location  $x_0 = 0$ , then solving Equation (4) for applied bi-harmonic signal  $f(t)$ , will result in a solution showing a single localised soliton wave as shown in Fig. 6 (a-b). In addition, the effect of changing the frequency of the perturbation  $f(t)$  can be clearly observed in Fig. 7(a-b) which is acquired for lower frequency component  $\omega_2 = 0.03$ . The soliton interaction with the external signal is further exemplified by the background waves getting stronger and this is very well demonstrated on the 2D projected plot (7b). This phenomenon becomes more interesting when changing the frequency values that are used to generate Fig. (8) for which  $\omega_1$  and  $\omega_2$  are 1.618033988 (the golden number) and 1, respectively. The main soliton gets less prominent as the interaction with the applied perturbation further intensifies as clearly shown in both (a) and (b). As a further exploration of how the inhomogeneity parameter  $a$  controls the resulting soliton wave evolution, we solve Equation (4) for  $a = 0.03$  with  $f(t)$  having  $\omega_1 = 0.056$  and  $\omega_2 = 0.03$ . Fig. (9) demonstrates a remarkable wave trend for which eminent breather solitons develop as depicted in (a) and (b).

**Discussion:** The data presented earlier show that changing the inhomogeneity parameter from 0 to 0.012 results in a solution of the INLSE that produces a rogue wave while evolving. Also, up to a

certain threshold of noise amplitude that is equal to 7, this wave is found to stand eminent in the presence of uniformly distributed noise. The interaction of the solitons with the external quasiperiodic field shows that the resulting wave trend depends mostly on the frequency of the applied perturbation. To a certain point, this interaction is also manifested in competition between the generated soliton and its background, depending on the quasi-periodicity of the field, which dictates the stability of the interacting solitons. Nonlinear stability of solitons against external perturbations was reported earlier in homogeneous NLSE<sup>15</sup>. The researchers found that larger amplitude solitons develop instability earlier than weaker amplitude ones, and this occurs for the same relative perturbation in play.

In addition, setting frequencies that are equal to the golden mean has impacted the wave pattern, enhanced the disorder, and resulted in less prominent solitons with the onset of deformation building in the main wave, as one may recognise in Fig. 7 and Fig 8. This scenario may be advocated if one considers here that both frequencies associated with the applied signal are incommensurate in nature and have a fixed value. This is of outstanding importance in dynamical systems especially when their ratio  $\omega_2/\omega_1$  is an irrational number, since it may give rise to erratic behaviour and open the route to chaos<sup>16,17,18</sup>, where in the presence of asymmetries and frequency incommensurability, massive phase fluctuations may occur. We stress that the soliton wave trend is crucially dependent on other system parameters too, such as the amplitude of the perturbation signal and inhomogeneity parameter, which dictate the overall spatiotemporal evolution of the rising soliton as observed. Driven by a phase-modulated excitation, the dynamics of cavity solitons (CS) has been investigated in a Kerr mediated passive optical fibre resonator<sup>19</sup>. The semi-analytical results allude to the complexity of how the solitons individually or mutually interact. In the absence of field, the two co-propagating solitons can attract, repulse or can propagate independently taking their initial delay into account. Whereas in

the presence of phase-modulated perturbation, the interaction of the two solitons may result in breathing, merging, annihilation, or two-soliton state, all depending on frequency and pump power. In addition, Ablowitz M J et al. studied the perturbations of dark solitons in nonlinear Schrödinger equation<sup>20</sup>. The problem was divided in two regions; inner region where the core of solitons exists, and outer region which evolves independently of the soliton. Their findings reported that a shelf around the soliton set in and propagated with speed governed by the background intensity. They also provided analysis for the background by considering both constant and slow evolution of which.

**Conclusion:** In this paper, an NLSE equation with inhomogeneity and quasiperiodicity terms has been discussed. Firstly, the spatiotemporal evolution of localised bright and dark solitons has been obtained by solutions to the NLSE which accounted for the homogeneous case. Then the soliton wave progressions have been studied for inhomogeneous terms  $a$  embedded in the system, which demonstrated that the wave intensity got higher as time went by for  $a = 0.012$ . In the presence of uniformly distributed noise with random seed and zero mean, the system markedly developed robust soliton waves with manifestly noisy peaks triggered in the background. Further, the system has been subject to an external applied perturbation with quasiperiodic frequencies. For  $\omega_1 = 0.056$  and  $\omega_2 = 1$ , solution to the INSLE yielded two solitons emanating from a rippling background. As a ramification to applying a Gaussian pulse wave at input, a pronounced and localised soliton with clearly ordered pattern has been generated for the same frequencies of the field. Upon reducing the frequency  $\omega_2$  to 0.03, the soliton developed background waves with notably increased intensity. In addition, the soliton-filed interaction for frequencies  $\omega_1 = 1.618033988$  (golden number) and  $\omega_2 = 1$  of incommensurate ratio has been considered to check the impact on the dynamics of the obtained solitons. The latter had demonstrated that the main soliton became less prominent with

background waves intensifying for the same Gaussian pulse applied to system. Finally, to study the impact of the nonuniformity parameter  $a$  on the soliton evolution, a higher value has been adopted ( $= 0.03$ ) with frequencies  $\omega_1, \omega_2$  0.056 and 0.03 respectively, which led to generating notably breather waves. The role of the system parameters which can be controlled has been addressed, in addition to the quasiperiodic perturbation field with emphasis on the contribution of incommensurability of frequencies on the soliton evolution and stability.

#### Competing interest

The author declares that the research was conducted in absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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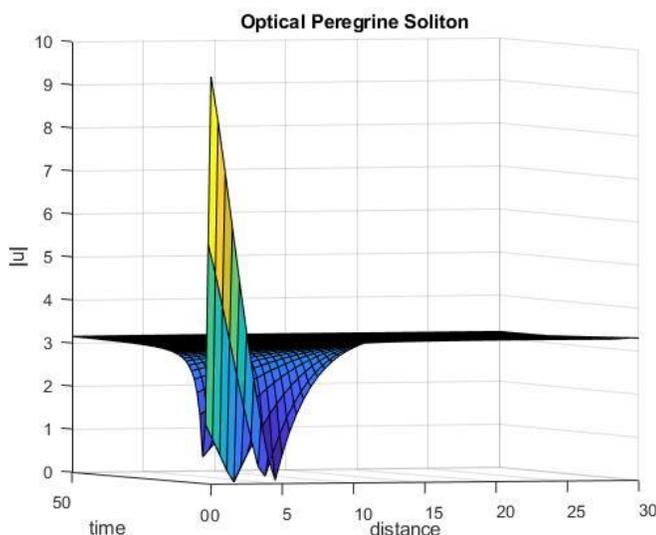
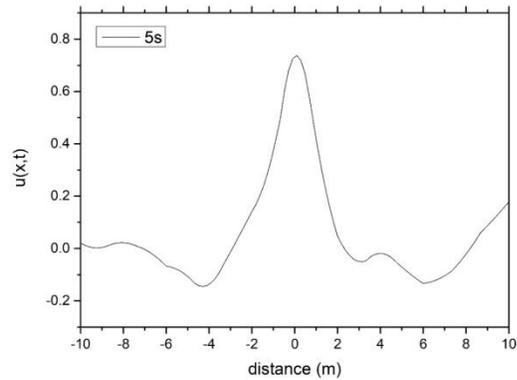
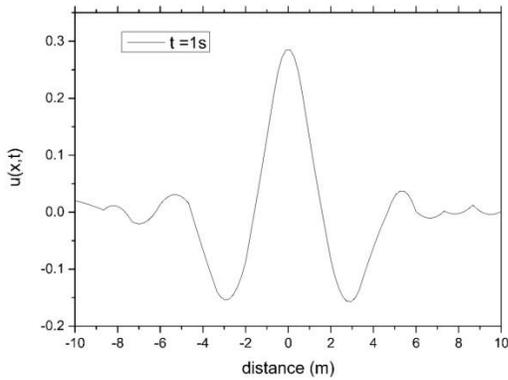
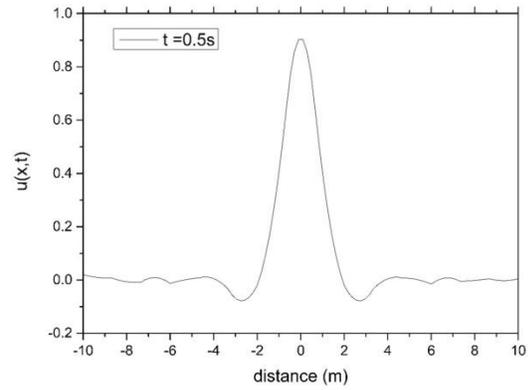
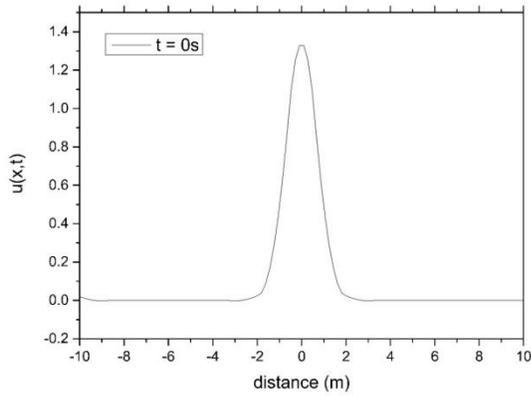


Figure 1. Optical Peregrine soliton wave resulting from solution to the NLSE.



Surface: -u (1)

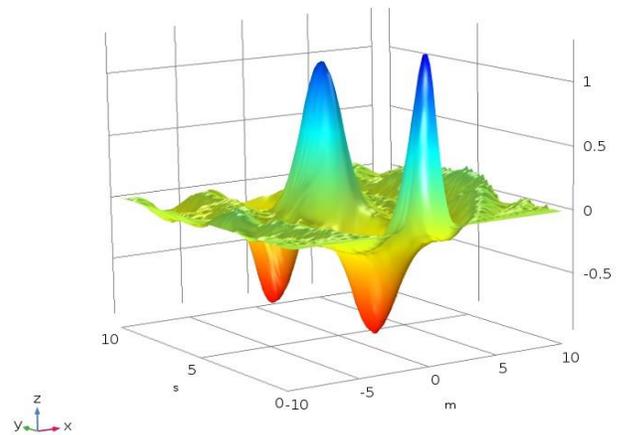
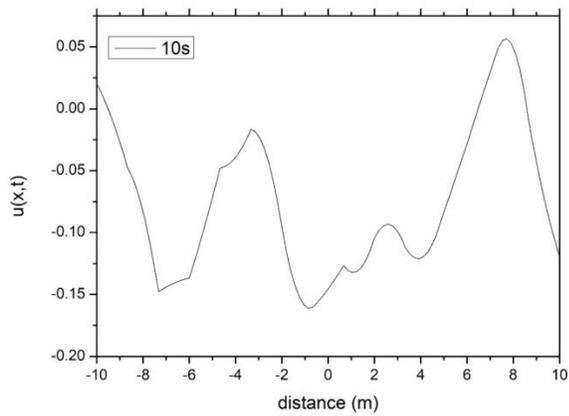


Fig 2 (c)

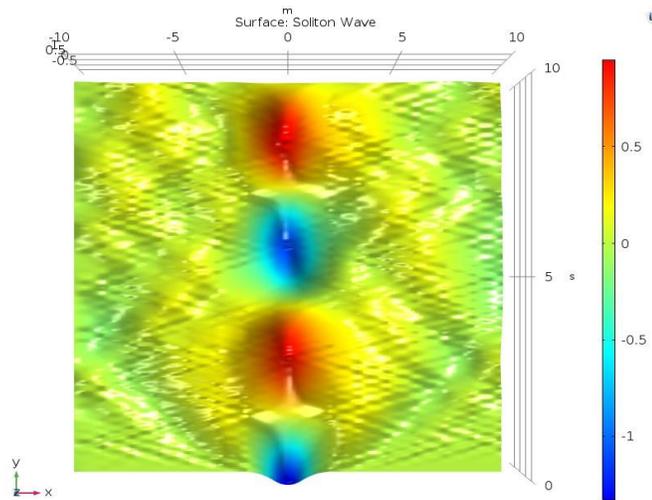


Figure 2. Localised bright and dark solitons of NLSE for the nonuniformity parameter  $a = 0$ : (a) temporal evolution snapshots, (b) 3D-plot of the localised solitons and (c) 2D-projected view showing the bright and dark soliton peaks with rippling background.

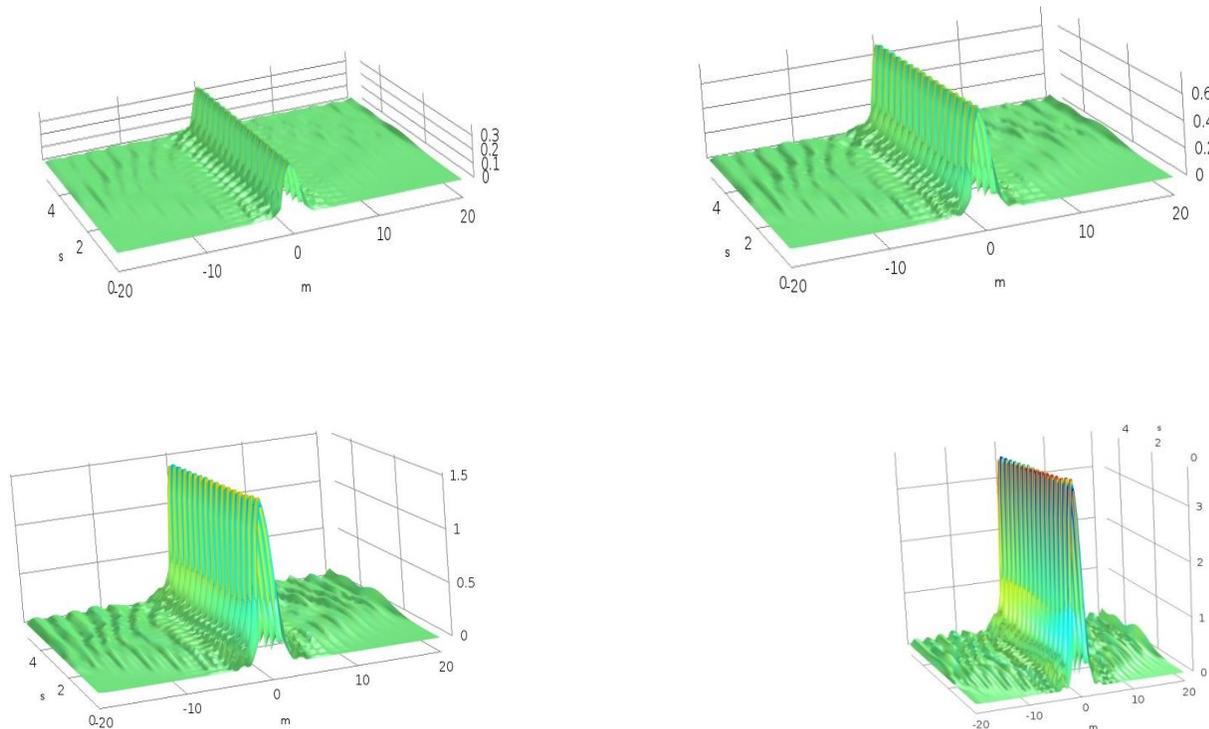


Figure 3. Soliton waves progression for inhomogeneity terms present in the NLSE,  $a = 0.012$ . From (a) to (d) the wave amplitude gets higher by time with soliton developing weak wave packet background.

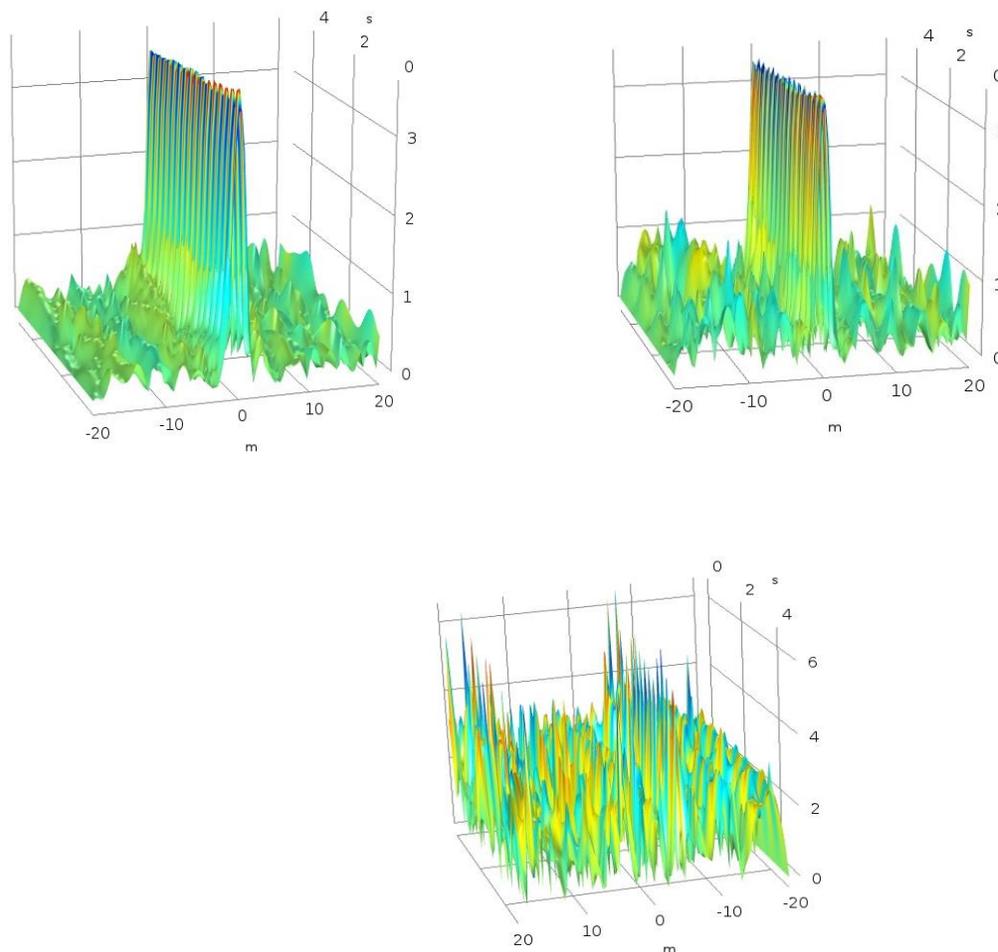


Figure 4. Robust rogue soliton wave for  $a = 0.012$  in presence of uniformly distributed noise with random seed and zero mean. Noise amplitude is equal to 1, 2 and 7 for (a), (b) and (c) respectively with manifestly noisy background starting to prevail for (c).

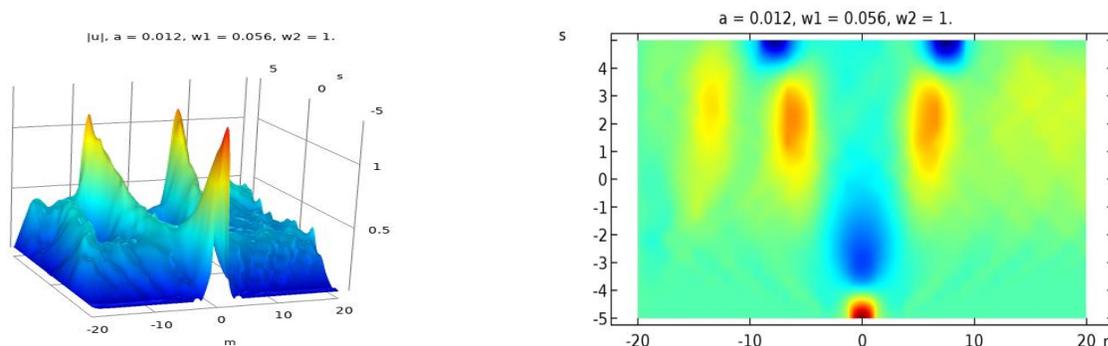
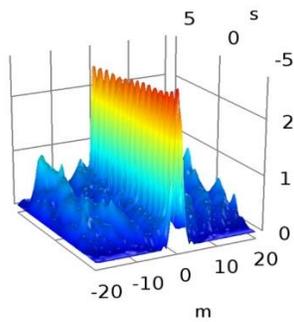


Figure 5. Soliton evolution under quasiperiodic perturbation for  $a = 0.012$ . (a) Two solitons generated under field frequencies  $\omega_1 = 0.056$  and  $\omega_2 = 1$  (b) 2D-top view plot of the emanating waves.

$|u|$ ,  $a = 0.012$ ,  $w1 = 0.056$ ,  $w2 = 1$ , Gaussian pulse.



$a = 0.012$ ,  $w1 = 0.056$ ,  $w2 = 1$ , Gaussian pulse.

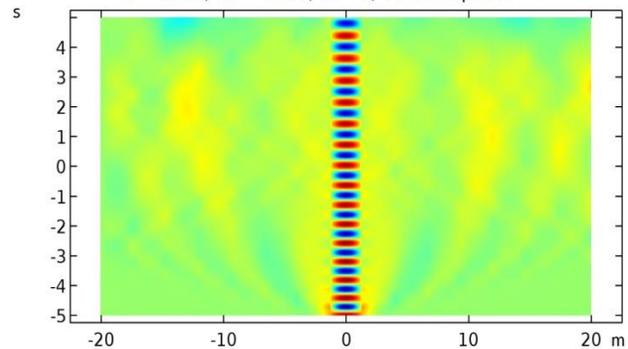
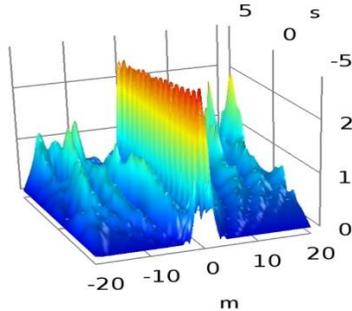


Figure 6. (a) Evolution of a pronounced localised soliton produced from solution to the INLSE for input Gaussian pulse wave at  $x_0 = 0$  under applied field  $f(t)$  (b) 2D-view of the localised soliton wave.

$|u|$ ,  $a = 0.012$ ,  $w1 = 0.056$ ,  $w2 = 0.03$ .



$a = 0.012$ ,  $w1 = 0.056$ ,  $w2 = 0.03$ .

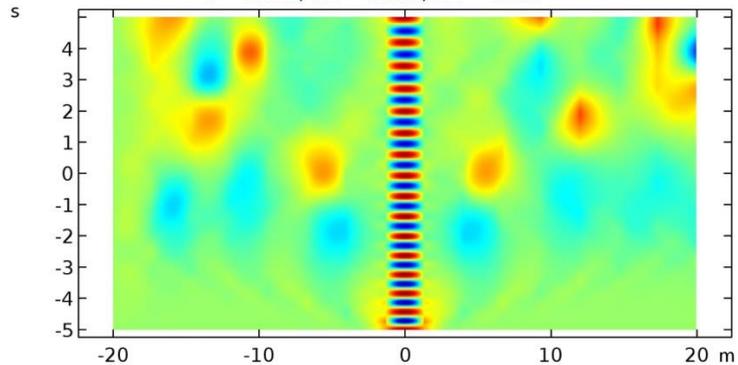
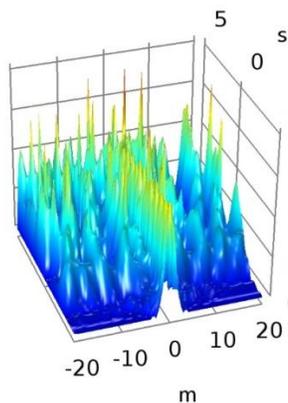


Figure 7. Effect of changing frequency on soliton wave (a) soliton interaction with field for lower frequency  $\omega_2 = 0.03$  developing background waves with increasing intensity (b) Surface view of soliton wave in (a).

$|u|$ ,  $a = 0.012$ ,  $w1 = 1.618033988$ ,  $w2 = 1$ , Gaussian pulse.



$a = 0.012$ ,  $w1 = 1.618033988$ ,  $w2 = 1$ , Gaussian pulse.

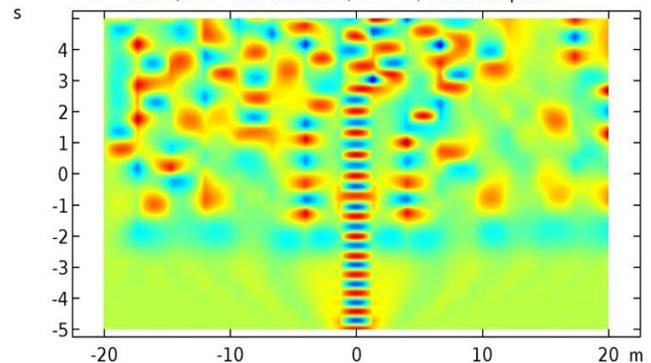


Figure 8. Soliton-field interaction for frequencies  $\omega_1$  and  $\omega_2$  are equal to 1.618033988 (golden number) and 1, respectively. (a) Main soliton becoming less prominent with background waves intensifying (b) Surface plot of soliton interacting with external perturbation.

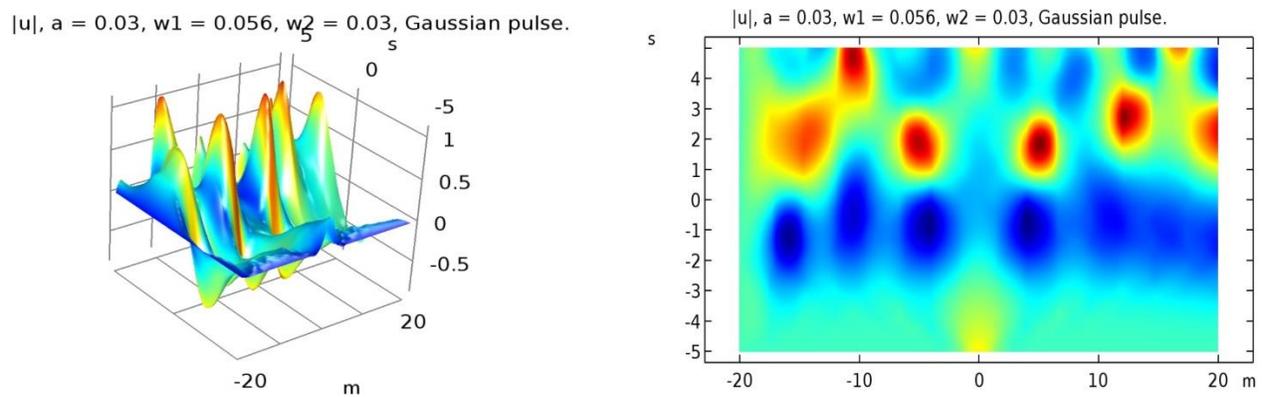


Figure 9. Influence of the nonuniformity parameter  $a$  as ‘control’ on the soliton dynamics (a) Breather waves evolution resulting for changing  $a$  to  $0.03$ , field frequencies depicted on picture (b) Surface plot of breather solitons.